LECTURE 2

WHITNEY'S EXTENSION THM (1934)

RECALL NOTATION

$F \in C^m(\mathbb{R}^n)$, $x \in \mathbb{R}^n \Longrightarrow$

$J_x(F) = m^{\underline{th}}$ degree Taylor poly of F at x

$$J_x(F)(y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha F(x) \cdot (y-x)^\alpha.$$

So

$J_x(F) \in \mathcal{P}$

$\mathcal{P} = \left[\begin{array}{l} \text{VECTOR SPACE OF ALL (REAL-VALUED)} \\ \text{POLYS OF DEGREE} \leq m \text{ on } \mathbb{R}^n \end{array} \right]$

$$C^m(\mathbb{R}^n) = \left\{ \begin{array}{l} \text{SPACE OF REAL-VALUED} \\ \text{FNS ON } \mathbb{R}^n \\ \text{WHOSE DERIVATIVES} \\ \text{UP TO ORDER } m \\ \text{ARE CONTINUOUS} \\ \text{AND BOUNDED} \end{array} \right\}$$

$$\|F\|_{C^m(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \le m} |\partial^\alpha F(x)|.$$

A <u>WHITNEY FIELD</u>

IS A FAMILY

$$\vec{P} = (P^x)_{x \in E}$$

of POLYNOMIALS $P^x \in \mathcal{P}$,

INDEXED BY THE POINTS $x \in E$,

WHERE E IS A SUBSET OF $\mathbb{R}^n$.

WHITNEY'S QUESTION:

GIVEN A WHITNEY FIELD

$$\vec{P} = (P^x)_{x \in E}$$

WITH $E \subset \mathbb{R}^n$ COMPACT.

HOW CAN WE DECIDE WHETHER THERE EXISTS $F \in C^m(\mathbb{R}^n)$ SUCH THAT

$$J_x(F) = P^x \text{ FOR ALL } x \in E \text{ ?}$$

COMPARE WITH QUESTION FROM TALK I :

GIVEN A FUNCTION $f : E \to \mathbb{R}$, HOW CAN WE DECIDE WHETHER THERE EXISTS $F \in C^m(\mathbb{R}^n)$ SUCH THAT

$$F(x) = f(x) \text{ FOR ALL } x \in E.$$

WHITNEY'S THM (1rst VERSION)

For a Whitney field $\vec{P} = (P^x)_{x \in E}$, E compact, the following are equivalent:

(A) There exists $F \in C^m(\mathbb{R}^n)$ s.t. $J_x(F) = P^x$ for all $x \in E$.

(B) For each multi-index α $(|\alpha| \leq m)$, we have
$$|\partial^\alpha(P^x - P^y)(x)| = o\left(|x-y|^{m-|\alpha|}\right)$$
as $|x-y| \to 0$, $x, y \in E$.

More Precise Version

Let $\vec{P} = (P^x)_{x \in E}$ be a Whitney fld.

Let M be a non-negative real number.

Suppose :

(A) $|\partial^\alpha P^x(x)| \leq M$ for all $|\alpha| \leq m$, $x \in E$.

(B) $|\partial^\alpha (P^x - P^y)(x)| \leq M |x-y|^{m-|\alpha|}$

for all $|\alpha| \leq m-1$, $x, y \in E$.

(C) $|\partial^\alpha (P^x - P^y)(x)| = o(|x-y|^{m-|\alpha|})$

as $|x-y| \to 0$ $(x, y \in E)$ for each $|\alpha| \leq m$.

Then $\exists\, F \in C^m(\mathbb{R}^n)$ s.t.

- $J_x(F) = P^x$ (all $x \in E$)
- $\|F\|_{C^m(\mathbb{R}^n)} \leq CM$ DEPENDS ONLY ON m, n

Thus, WE CAN COMPUTE

THE LEAST POSSIBLE (inf)

C^m NORM OF A FUNCTION F

that agrees with a given

Whitney field $\vec{P} = (P^x)_{x \in E}$

UP TO A CONST FACTOR

DEPENDING ONLY ON m, n.

FOR $C^2(\mathbb{R}^n)$

(WITH A BETTER CHOICE OF THE C^2 NORM),

LEGRUYER & WELLS

FOUND AN <u>EXACT FORMULA</u>

for the least possible norm!

WHITNEY'S PROOF GIVES AN EXPLICIT FORMULA FOR F.

F DEPENDS LINEARLY ON $\vec{P}$

IF $\vec{P} = (P^x)_{x \in E}$, THEN

FOR ANY GIVEN POINT $y \in \mathbb{R}^n$,

$F(y)$ IS DETERMINED ENTIRELY

BY $P^{x_1}, \dots, P^{x_k}$.

- $K \leq C$ ← DEPENDS ONLY ON m, n
- $x_1, \dots, x_k$ DEPEND ON y BUT NOT ON $\vec{P}$.

$$\vec{P} = (P^x)_{x \in E}, \quad F = T\vec{P} \in C^m(\mathbb{R}^n)$$

$$\Longrightarrow$$

$$\boxed{F(y) = \sum_{k=1}^{K} \lambda_k (P^{x_k})}$$

$\lambda_k : \mathcal{P} \longrightarrow \mathbb{R}$ ARE LINEAR FUNCTIONALS

$x_1, \dots, x_K \in E$

$K \leq C$

The λ_k and x_k DEPEND ON y, BUT NOT ON $\vec{P}$.

$\vec{P} \longmapsto F$ HAS "BOUNDED DEPTH"

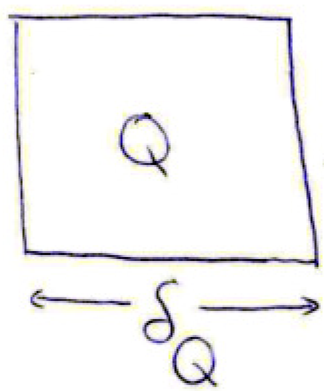
SKETCH OF

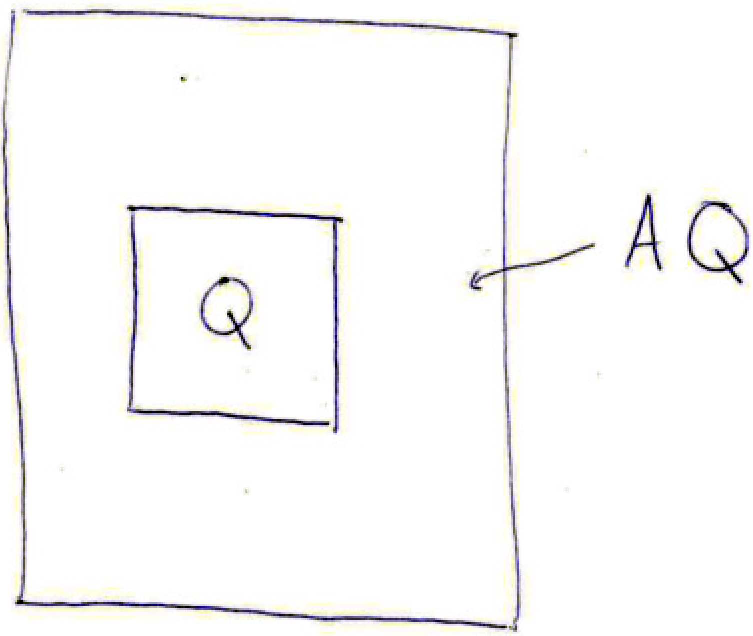
WHITNEY'S PROOF

MAIN STEPS

- WHITNEY CUBES
- WHITNEY PARTITION OF UNITY
- THE EXTENSION F
- CHECK THAT IT WORKS

PREPARE TO DEFINE WHITNEY CUBES

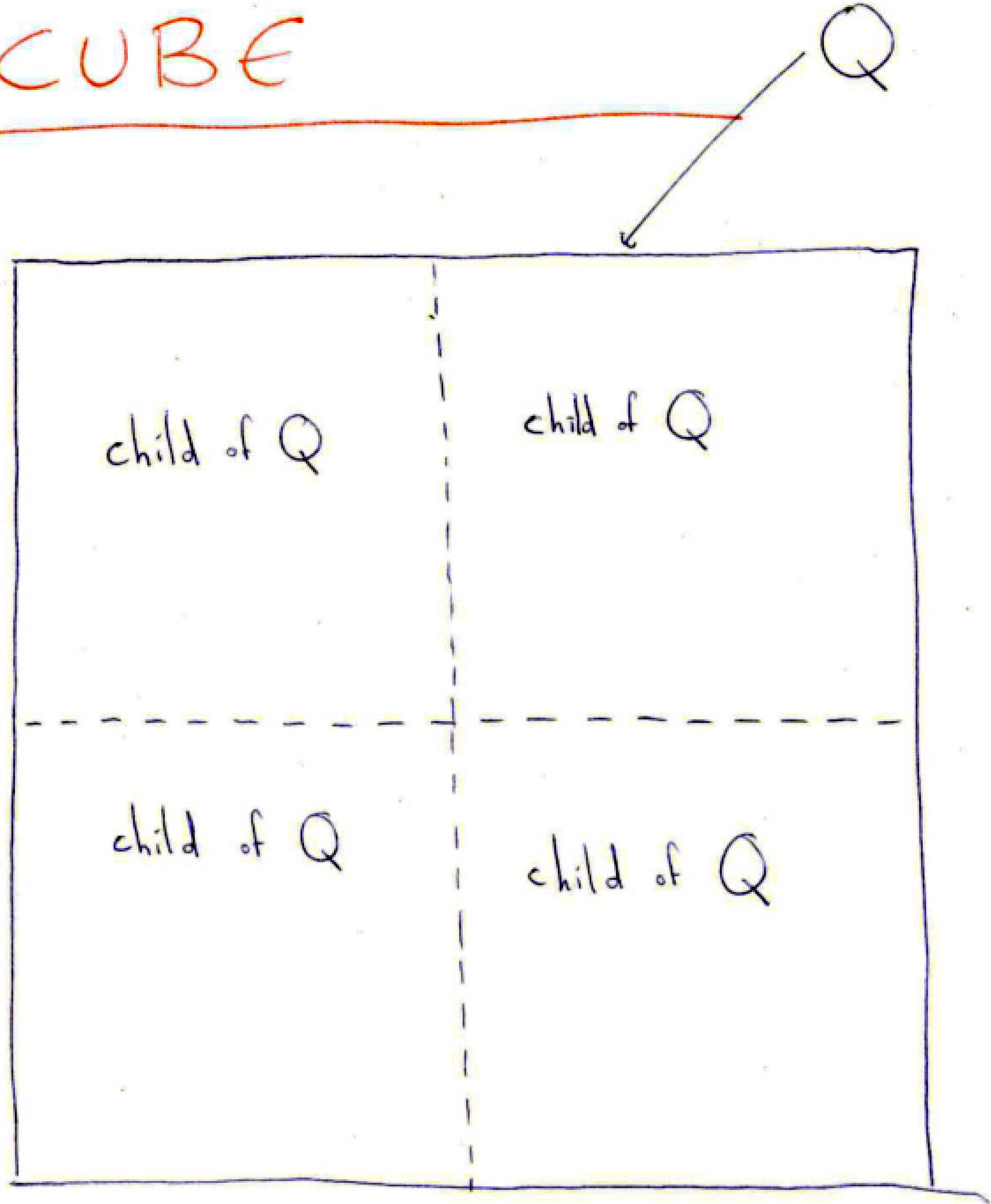




- $\delta_{AQ} = \delta_Q$

- AQ & Q HAVE SAME CENTER

BISECTING A CUBE

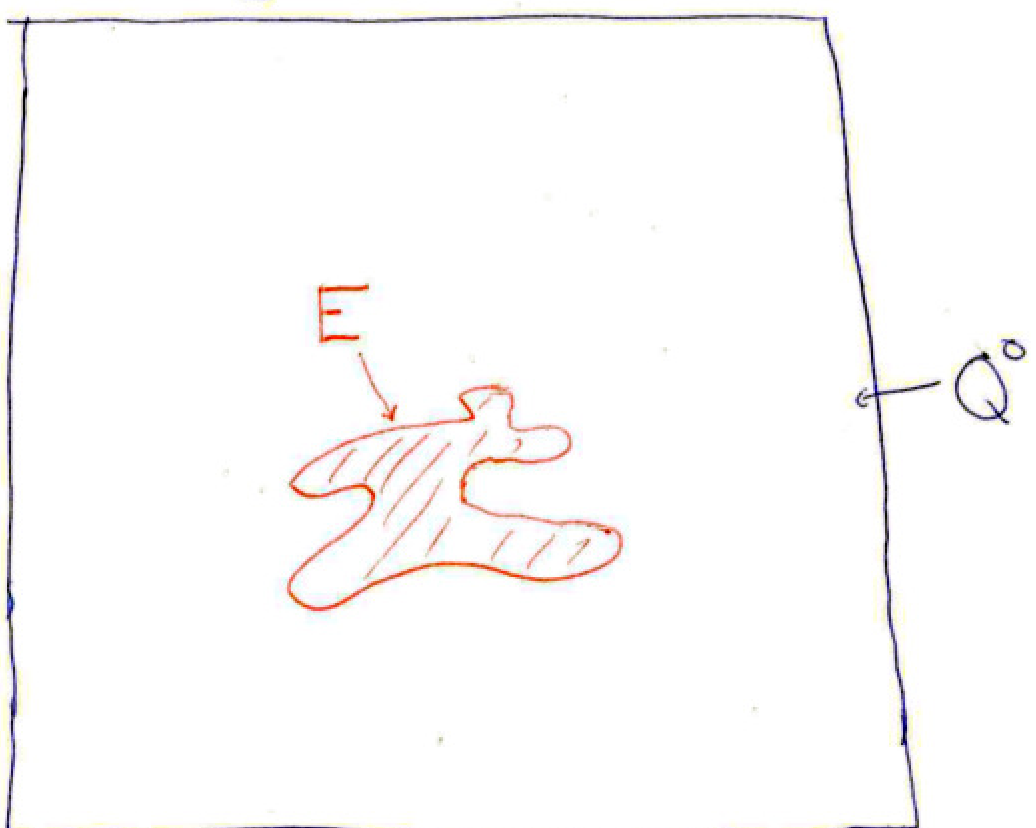


CONSTRUCTING WHITNEY CUBES

Given $E \subset \mathbb{R}^n$.

Start with a big cube Q^0.

containing E in its middle half



$\left(\text{SAY } \delta_{Q^0} = 1024 \right)$

CONSTRUCTION OF THE WHITNEY CUBES

PROCEEDS IN STEPS:

STEP 0,

STEP 1,

STEP 2,

$\vdots$

AT EACH STEP i

WE PRODUCE A PARTITION

OF Q^0 INTO FINITELY

MANY CUBES.

THE PARTITION AT STEP 0

CONSISTS OF THE SINGLE

CUBE Q^0.

THE PARTITION IN STEP $(i+1)$ REFINES THE PARTITION IN STEP i.

TO PRODUCE THE PARTITION OF STEP $(i+1)$, WE BISECT SOME OF THE CUBES FROM THE PARTITION IN STEP i

RULE:

LET Q BE A CUBE OF THE STEP i PARTITION.

- IF $3Q \cap E = \emptyset$, THEN WE INCLUDE Q IN THE STEP $(i+1)$ PARTITION, AND WE CALL Q A

$\boxed{\text{WHITNEY CUBE}}$

- IF $3Q \cap E \neq \emptyset$,

THEN WE BISECT Q

INTO ITS CHILDREN

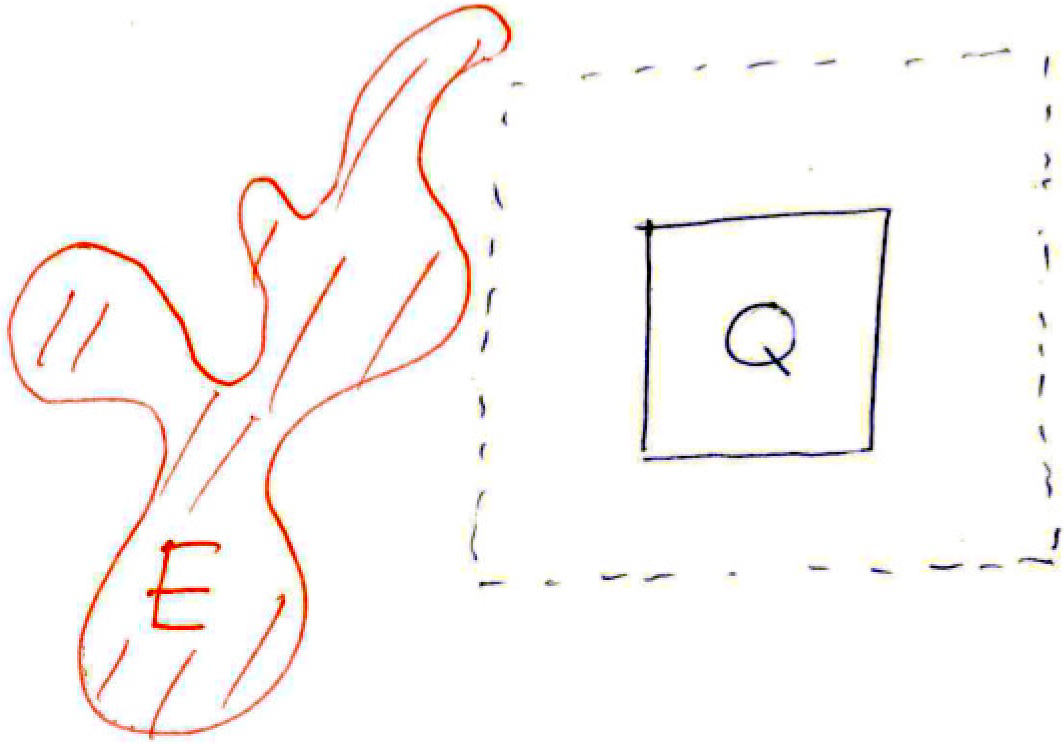
$Q_1, Q_2, \cdots, Q_{2^n}$,

AND INCLUDE THOSE

CHILDREN (BUT NOT Q)

IN THE STEP $(i+1)$

PARTITION.



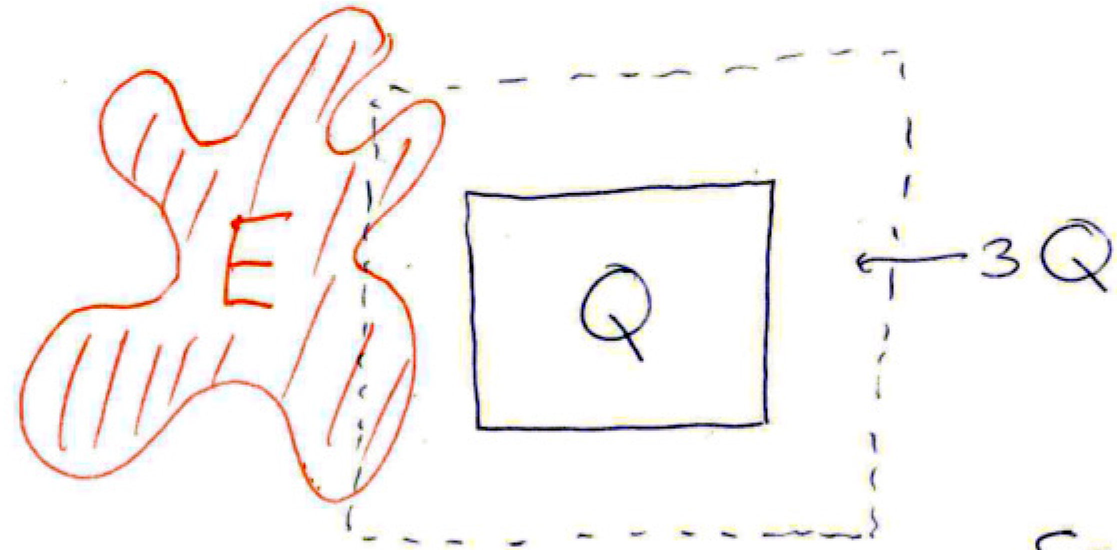
Q ARISES AT STEP i

$3Q \cap E = \emptyset$

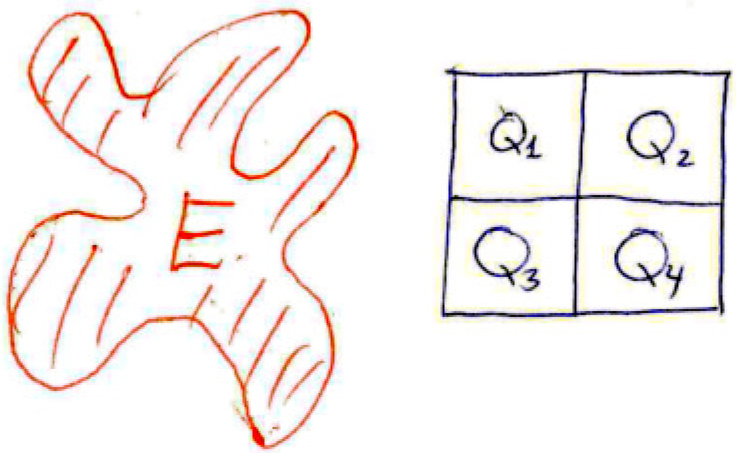
$\Longrightarrow$ Q SURVIVES, AND

APPEARS AT STEP $i+1$

$\boxed{\text{THIS } Q \text{ IS A WHITNEY CUBE}}$



STEP i



STEP $i+1$

Q ARISES AT STEP i,

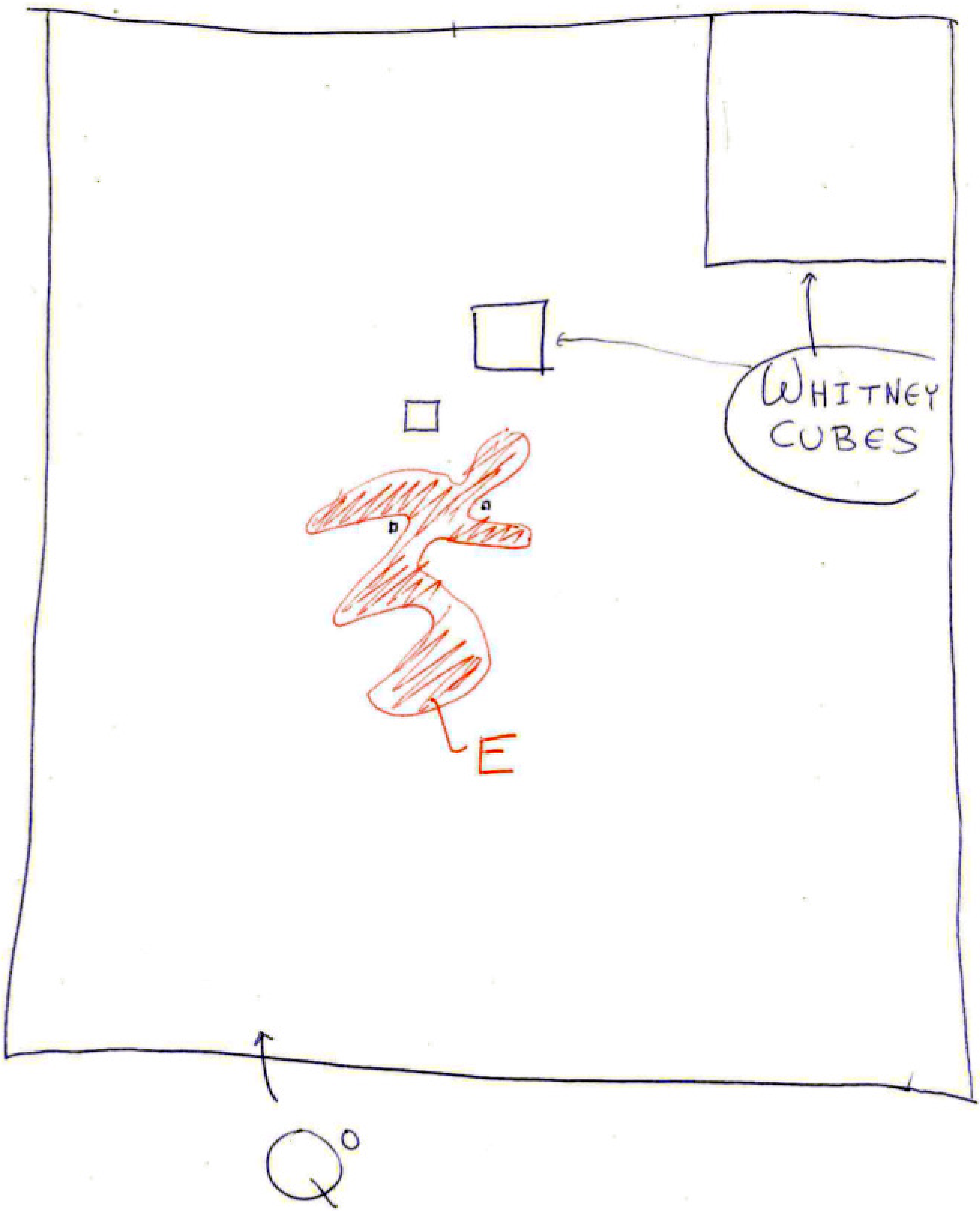
$3Q \cap E \neq \emptyset \Longrightarrow$

AT STEP $(i+1)$, Q IS REPLACED BY ITS 4 CHILDREN $Q_1, \dots, Q_4$.

BASIC PROPERTIES OF WHITNEY CUBES

- THE WHITNEY CUBES FORM A PARTITION OF $\mathbb{R}^n \setminus E$.

- THE DIAMETER OF ANY WHITNEY CUBE IS COMPARABLE TO ITS DISTANCE FROM E.


WHITNEY CUBES
a
b
E
Q°

PROOF :

Let Q = Whitney cube.

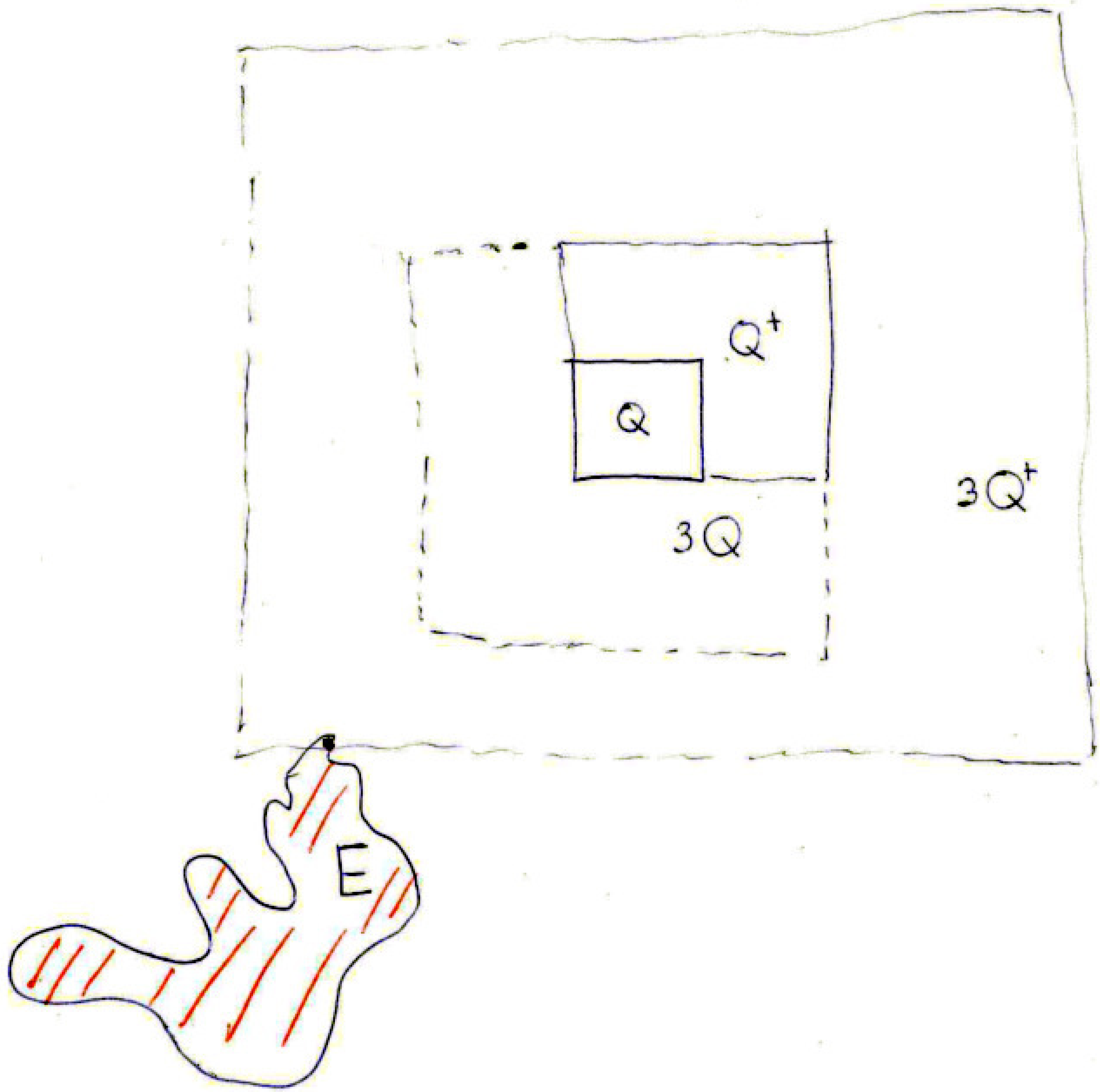
Q arose as a child of a cube Q^+.

We decided to retain Q,

but we decided <u>not</u> to retain Q^+.

So $\quad 3Q \cap E = \phi$

but $\quad 3Q^+ \cap E \neq \phi$.



Q^+
Q
$3Q^+$
$3Q$
E

- IF TWO WHITNEY CUBES Q & Q' TOUCH, THEN δ_Q & $\delta_{Q'}$ ARE COMPARABLE,

$$\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q .$$

- ANY GIVEN WHITNEY CUBE Q TOUCHES AT MOST A BOUNDED NUMBER OF OTHER WHITNEY CUBES Q'!

WE HAVE DEFINED THE WHITNEY CUBES.

NEXT, WE DEFINE THE WHITNEY PARTITION OF UNITY

FOR EACH WHITNEY CUBE Q,

FIX A SMOOTH FUNCTION $\tilde{\theta}_Q$

ON $\mathbb{R}^n$, SATISFYING:

- $\tilde{\theta}_Q \geqslant 0$
- $\tilde{\theta}_Q = 1$ on Q
- $\tilde{\theta}_Q$ SUPPORTED IN $(1.01)Q$
- $|\partial^\alpha \tilde{\theta}_Q| \leq C \delta_Q^{-|\alpha|}$ FOR $|\alpha| \leq m$

Let $\psi = \sum_{Q'} \tilde{\theta}_{Q'}$, WHERE Q'

VARIES OVER ALL THE WHITNEY CUBES.

If $x \in \mathrm{Supp}(\tilde{\theta}_{Q'})$, then $\delta_{Q'} \sim \mathrm{dist}(x, E)$.

In particular, $x \in \mathrm{supp}(\tilde{\theta}_{Q'})$ for at most C distinct Whitney cubes Q'.

Therefore, ψ has the following properties.

$$\psi = \sum_{Q'} \tilde{\theta}_{Q'}$$

- $\psi \geq 1$ on $\mathbb{R}^n \setminus E$
- $\psi = 0$ on E
- $|\partial^\alpha \psi(x)| \leq C\,[\mathrm{dist}(x, E)]^{-|\alpha|}$

  For $|\alpha| \leq m$, $x \in \mathbb{R}^n \setminus E$

$$\boxed{\begin{array}{l} \bullet\ |\partial^\alpha \psi(x)| \leq C\,\delta_Q^{-|\alpha|} \quad \text{on } \operatorname{supp}(\tilde{\theta}_Q) \\ \bullet\ \psi(x) \geq 1 \quad \text{on } Q. \end{array}}$$

Now, for each Whitney cube Q, we set

$$\theta_Q = \frac{\tilde{\theta}_Q}{\psi} = \frac{\tilde{\theta}_Q}{\sum_{Q'} \tilde{\theta}_{Q'}} \quad \underline{\underline{\text{on } Q^\circ}}$$

Thus,

$$\sum_Q \theta_Q = \begin{cases} 1 & \text{for } x \in Q^\circ \setminus E \\ 0 & \text{for } x \in E \end{cases}.$$

This is the

<u>Whitney Partition of Unity</u>

FROM THE PROPERTIES OF $\tilde{\theta}_Q$, ψ, WE EASILY DERIVE THE BASIC PROPERTIES OF THE θ_Q.

BASIC PROPERTIES OF θ_Q

- $$\sum_Q \theta_Q(x) = \begin{cases} 1 & \text{FOR } x \in Q^\circ \setminus E \\ 0 & \text{FOR } x \in E \end{cases}$$

FOR EACH WHITNEY CUBE Q,

- $\operatorname{supp}(\theta_Q) \subset (1.01)\,Q$

and

- $|\partial^\alpha \theta_Q(x)| \leq C\,\delta_Q^{-|\alpha|}$ FOR ALL $x \in Q^\circ$, $|\alpha| \leq m$.

WE HAVE NOW DEFINED THE

WHITNEY CUBES

and the

WHITNEY PARTITION OF UNITY.

NOW WE CAN DEFINE

THE FUNCTION F.

FOR EACH WHITNEY CUBE Q,
WE PICK A POINT $x(Q) \in E$
AS CLOSE AS POSSIBLE TO E.

RECALL, WE ARE GIVEN A
WHITNEY FIELD

$$\vec{P} = (P^x)_{x \in E}$$

WE WANT A FUNCTION $F \in C^m(\mathbb{R}^n)$
s.t.
$J_x(F) = P^x$ FOR EACH $x \in E$.

WILL DEFINE F ON Q°

DEFINE

$$F(x) = \begin{cases} \sum_Q \theta_Q(x)\, P^{x(Q)}(x) & \text{FOR } x \in Q^\circ \setminus E \\ P^x(x) & \text{FOR } x \in E \end{cases}$$

NOTE: F DEPENDS LINEARLY ON $\vec{P} = (P^x)_{x \in E}$ IN A VERY SIMPLE WAY.

The map $\vec{P} \longmapsto F$ is a linear map of bounded depth.

NOW WE HAVE DEFINED
THE FUNCTION F

MUST SHOW

$$\left\{ \begin{array}{c} \text{HYPOTHESES} \\ \text{OF} \\ \text{WHITNEY'S THM.} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} F \in C^m(Q^0) \\ J_x(F) = P^x \quad (x \in E) \\ \|F\|_{C^m(Q^0)} \leq CM \end{array} \right.$$

RECALL HYPOTHESES:

- $|\partial^\alpha P^x(x)| \leq M$ (all $x \in E$, $|\alpha| \leq m$)

- $|\partial^\alpha (P^x - P^y)(x)| \leq M |x-y|^{m-|\alpha|}$

  $(x, y \in E, \ |\alpha| \leq m-1)$

- $|\partial^\alpha (P^x - P^y)(x)| = o\left(|x-y|^{m-|\alpha|}\right)$

  as $|x-y| \to 0$, $x, y \in E$, $|\alpha| \leq m$.

KEY IDEA

Let $x \in Q^\circ \setminus E$.

Say $x \in \hat{Q}$ (WHITNEY CUBE).

Then in a nbd. of x, write

$$F = \sum_Q \theta_Q P^{x(Q)}$$

$$= P^{x(\hat{Q})} + \sum_Q \theta_Q \cdot [P^{x(Q)} - P^{x(\hat{Q})}].$$

TO SHOW THAT

$$\|F\|_{C^m(Q^\circ)} \leq CM,$$

IT WILL BE ENOUGH

TO SHOW THAT

$$\boxed{\begin{gathered}\left|\partial^\alpha\left\{\theta_Q\cdot\left[P^{x(Q)}-P^{x(\hat{Q})}\right]\right\}(x)\right|\leq CM\\[1em] \text{FOR } \operatorname{supp}\theta_Q \ni x.\end{gathered}}$$

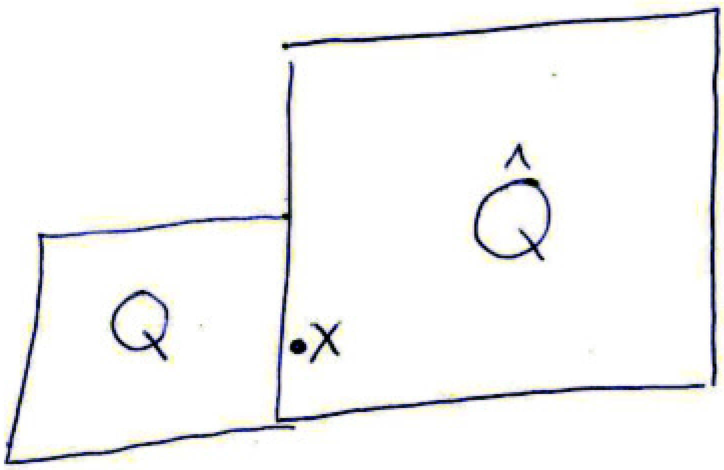
CRUCIAL POINT:

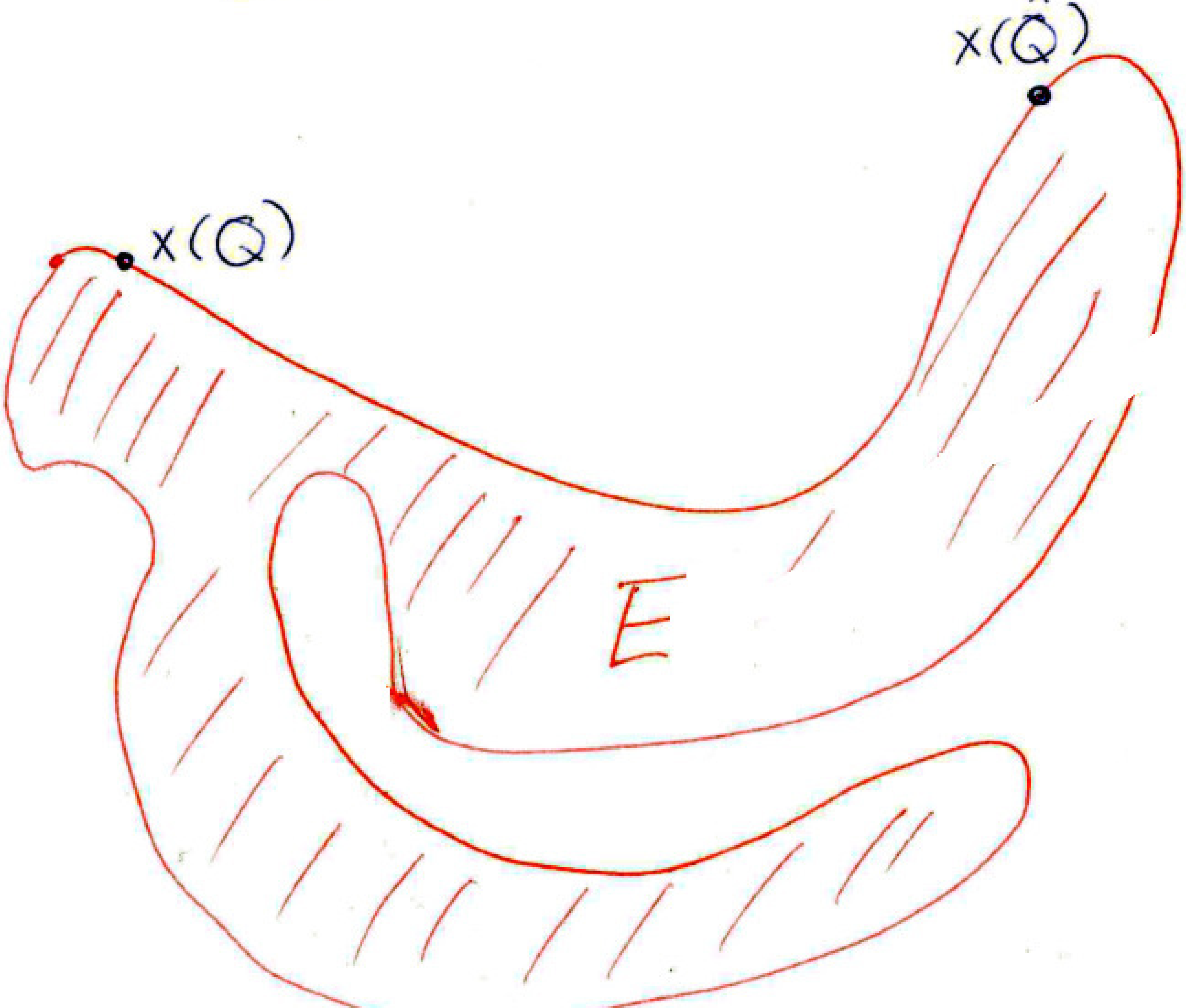
$$|x(Q) - x(\hat{Q})| \leq C\delta_{\hat{Q}}$$

AND

$$\tfrac{1}{2}\delta_{\hat{Q}} \leq \delta_Q \leq 2\delta_{\hat{Q}}$$

WHEN $\quad \operatorname{supp}\theta_Q \ni x$.


$\hat{Q}$
Q
x


$x(\hat{Q})$
$x(Q)$
E

TO ESTIMATE

$$\partial^\alpha \left\{ \theta_Q \cdot \left[P^{x(Q)} - P^{x(\hat{Q})} \right] \right\} (x)$$

WE USE THE ESTIMATES

$$\boxed{\begin{gathered} \left| \partial^\beta \theta_Q(x) \right| \leq C \delta_Q^{-|\beta|} \leq C' \delta_{\hat{Q}}^{-|\beta|} \\ \text{for } |\beta| \leq m \end{gathered}}$$

AND

$$\boxed{\begin{gathered} \left| \partial^\beta \left[P^{x(Q)} - P^{x(\hat{Q})} \right] (x) \right| \leq \\ CM \delta_{\hat{Q}}^{m-|\beta|} \quad \text{for } |\beta| \leq m. \end{gathered}}$$

$$\partial^\alpha \left\{ \theta_Q \cdot \left[P^{x(Q)} - P^{x(\hat{Q})} \right] \right\}(x)$$

IS A SUM (OVER $\beta + \gamma = \alpha$)

OF TERMS

$$\left[\partial^\beta \theta_Q(x) \right] \cdot \left[\partial^\gamma \left(P^{x(Q)} - P^{x(\hat{Q})} \right)(x) \right]$$

DOMINATED BY $\delta_{\hat{Q}}^{-|\beta|}$

DOMINATED BY $M \delta_{\hat{Q}}^{m-|\gamma|}$

PRODUCT $\lesssim M \delta_{\hat{Q}}^{m-|\alpha|}$

RECALL,

$$\hat{Q} \subset Q^{\circ} \longleftarrow \text{SIDELENGTH } 1024$$

SO $\delta_{\hat{Q}} \leq 1024.$

SO, FOR $|\alpha| \leq m$, WE HAVE SHOWN THAT

$$\left| \partial^{\alpha} \left\{ \theta_{Q} \cdot \left[P^{x(Q)} - P^{x(\hat{Q})} \right] \right\} (x) \right| \leq CM.$$

THEREFORE,

$$\| F \|_{C^{m}(Q^{\circ})} \leq CM.$$

Those are the Main Ideas

in the Proof of the

Whitney Extension Theorem

WHITNEY'S THEOREM TELLS US WHEN THERE EXISTS $F \in C^m$ THAT AGREES WITH A GIVEN WHITNEY FIELD ON E.

WE REALLY WANT TO KNOW WHETHER THERE EXISTS $F \in C^m$ THAT AGREES WITH A GIVEN FUNCTION on E.

Although WHITNEY'S THM

ANSWERS AN EASIER VARIANT

OF THE "REAL" PROBLEM,

BOTH THE THEOREM AND

ITS PROOF CONTAIN IMPORTANT

LESSONS FOR US

(& for ANALYSIS).

Lessons from the Proof

of Whitney's Theorem

LESSON 1

- WE WILL BE INTERESTED IN PRODUCTS OF THE FORM

$$[\text{FACTOR } 1] \cdot [\text{FACTOR } 2]$$

WHERE

$$\boxed{\begin{array}{l} |\partial^\beta(\text{FACTOR } 1)(x)| \le \delta^{-|\beta|} \\ \text{and} \\ |\partial^\beta(\text{FACTOR } 2)(x)| \le \delta^{m-|\beta|} \\ \text{for } |\beta| \le m. \end{array}}$$

Lesson 2

Whitney's Theorem

for

Finite Sets.

GIVEN $M \geq 0$, $\vec{P} = (P^x)_{x \in E}$,

E FINITE. ASSUME:

$$\boxed{\begin{array}{l} |\partial^\alpha P^x(x)| \leq M \quad \text{FOR ALL } x \in E,\ |\alpha| \leq m. \\[2ex] |\partial^\alpha (P^x - P^y)(x)| \leq M|x-y|^{m-|\alpha|}, \quad \text{ALL } x, y \in E, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad |\alpha| \leq m-1. \end{array}}$$

THEN THERE EXISTS

$F \in C^m(\mathbb{R}^n)$ WITH NORM

$$\boxed{\begin{array}{l} \|F\|_{C^m(\mathbb{R}^n)} \leq CM, \quad \text{SUCH THAT} \\[2ex] J_x(F) = P^x \quad \text{FOR ALL } x \in E \end{array}}$$

<u>NOTE</u> : TO DECIDE WHETHER

$$\boxed{\begin{gathered} |\partial^\alpha P^x(x)| \le M \\ \text{and} \\ |\partial^\alpha (P^x - P^y)(x)| \le M\,|x-y|^{m-|\alpha|}, \end{gathered}}$$

WE MAY EXAMINE

THE QUADRATIC FORM

$$\boxed{\mathcal{Q}(\vec{P}) = \sum_{x \in E} \sum_{|\alpha| \le m} \left(\partial^\alpha P^x(x)\right)^2 + \sum_{\substack{x,y \in E \\ (x \ne y)}} \sum_{|\alpha| \le m-1} \left(\frac{\partial^\alpha (P^x - P^y)(x)}{|x-y|^{m-|\alpha|}} \right)^2 .}$$

THE QUADRATIC FORM

$Q(\vec{P})$

IS A USEFUL IDEA

WHEN $\#(E) \leq C$,

BUT NOT WHEN $\#(E)$

IS ARBITRARILY LARGE.

APPLICATION

LET $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$ FINITE.

DEFINE

$$\|f\|_E = \boxed{\begin{array}{l} \text{INF OF } \|F\|_{C^m(\mathbb{R}^n)} \\ \text{OVER ALL } F \in C^m(\mathbb{R}^n) \\ \text{SUCH THAT } F = f \text{ on } E. \end{array}}$$

PROBLEM:

COMPUTE THE ORDER OF MAGNITUDE

OF

$\|f\|_E$.

SOLUTION IN CASE $\#(E) \leq C$

$\|f\|_E^2$ IS COMPARABLE TO

THE MIN OF THE QUADRATIC FORM

$\mathcal{Q}(\vec{P})$

OVER ALL $\vec{P} = (P^x)_{x \in E}$

SUCH THAT $P^x(x) = f(x)$ (ALL $x \in E$).

That's immediate from

Whitney's Thm for finite sets.

So, COMPUTING THE ORDER OF

MAGNITUDE OF $\|f\|_E$

WHEN $\#(E) \leq C$

IS REDUCED TO LINEAR ALGEBRA.

(& the matrices are of bounded size.)

Lesson 3

The Calderón - Zygmund Decomposition

START WITH A UNIT CUBE $Q^0 \subset \mathbb{R}^n$.

IF WE LIKE IT, THEN KEEP IT;

IF WE DON'T LIKE IT, THEN BISECT IT INTO SUBCUBES $Q_1, \ldots, Q_{2^n}$, & EXAMINE EACH OF THOSE IN TURN.

TO EXAMINE A CUBE Q,

WE ASK:

DO WE LIKE Q?

- If SO, THEN WE KEEP Q.
- If not, THEN WE BISECT Q

INTO 2^n SUBCUBES, AND

EXAMINE EACH OF

THOSE SUBCUBES.

STARTING WITH Q^0,

AND REPEATEDLY BISECTING

AS PRESCRIBED ABOVE,

WE ARRIVE AT A COLLECTION

OF PAIRWISE DISJOINT

SUBCUBES OF Q^0.

THEY ARE THE

CALDERÓN-ZYGMUND CUBES

HOW DO WE DECIDE WHETHER WE LIKE A CUBE Q?

WHITNEY (1934): We like Q if

$$3Q \cap E = \phi.$$

CALDERÓN & ZYGMUND (1950's; see also MARCINKIEWICZ, 1930's)

We like Q if

$$\frac{1}{\mathrm{vol}(Q)} \int_Q |f(x)|\, dx > \lambda$$

for given fn. f & number λ.

We can give any rule we please.

For each Calderón - Zygmund cube Q,

we know that

WE LIKE Q,

but

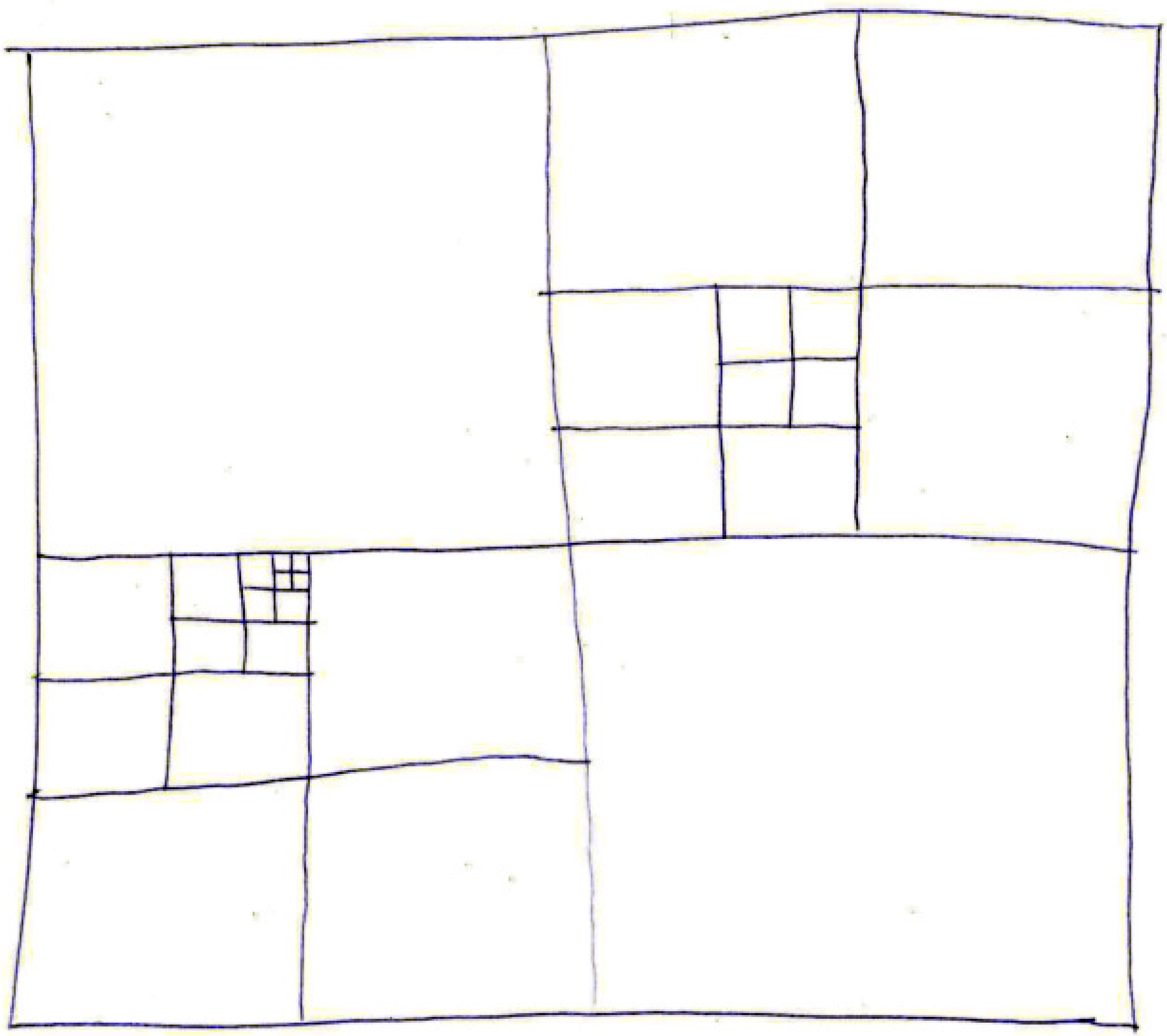
Q ARISES AS A CHILD OF ANOTHER CUBE Q^+, AND WE DON'T LIKE Q^+.

THE CALDERÓN-ZYGMUND CUBES ARE PAIRWISE DISJOINT.

IF WE LIKE EVERY SUFFICIENTLY SMALL CUBE,

THEN THE CALDERÓN-ZYGMUND CUBES PARTITION

$$Q^0.$$



$$\boxed{\text{THE CALDERÓN - ZYGMUND DECOMPOSITION IS A VERY IMPORTANT IDEA, WITH MANY APPLICATIONS.}}$$

WE WILL USE A PARTICULAR C-Z DECOMPOSITION TO PROVE OUR MAIN RESULTS.

AN APPLICATION OF THE WHITNEY CUBES

WE WILL CONSTRUCT THE "WELL-SEPARATED PAIRS DECOMPOSITION"

MOTIVATION:

Let $f : E \to \mathbb{R}$,

$E \subset \mathbb{R}^n$, $\#(E) = N$.

Want to compute the Lipschitz constant

$$\|f\|_{Lip} := \max_{\substack{x, y \in E \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

How many computer ops does it take?

Obvious method $\Rightarrow$ $\sim N^2$ ops.

We can do much better!

We can compute $\|f\|_{Lip}$ to within a 1% error using only $O(N \log N)$ computer ops.

Here's the idea...

SUPPOSE $E', E'' \subset E$ ARE

WELL-SEPARATED,

i.e.,

$$\text{DISTANCE}(E', E'') \geq 10^3 \cdot \left[\text{DIAM}(E') + \text{DIAM}(E'')\right]$$

FOR SUCH <u>WELL-SEPARATED</u> E', E'',

LET'S COMPUTE

$$\mathrm{MAX}\left\{ \frac{|f(x')-f(x'')|}{|x'-x''|} : (x',x'') \in E' \times E'' \right\}$$

NAÏVE METHOD $\Rightarrow$
USE $(\# E') \times (\# E'')$ COMPUTER OPS.

WE CAN DO MUCH BETTER,
THANKS TO TWO SIMPLE REMARKS.

FIRST REMARK:

$|x' - x''|$ IS ESSENTIALLY CONSTANT AS (x', x'') VARIES OVER $E' \times E''$.

THEREFORE, IT'S ENOUGH TO COMPUTE

$$\max\{\,|f(x') - f(x'')| : (x', x'') \in E' \times E''\,\}.$$

SECOND REMARK:

$$\text{MAX}\left\{ |f(x') - f(x'')| : (x', x'') \in E' \times E'' \right\}$$

IS EQUAL TO THE LARGER OF

THE TWO NUMBERS

$$\left[\max_{x' \in E'} f(x') \right] - \left[\min_{x'' \in E''} f(x'') \right]$$

AND

$$\left[\max_{x'' \in E''} f(x'') \right] - \left[\min_{x' \in E'} f(x') \right]$$

THEREFORE, WHEN E' & E'' ARE WELL-SEPARATED,

WE CAN COMPUTE

$$\text{MAX}\left\{ \frac{|f(x')-f(x'')|}{|x'-x''|} : (x', x'') \in E' \times E'' \right\}$$

TO WITHIN 1%,

USING $\sim (\# E') + (\# E'')$ COMPUTER OP'S,

A BIG IMPROVEMENT OVER

$(\# E') \times (\# E'')$.

This motivates the

WELL-SEPARATED PAIRS DECOMPOSITION

("WSPD")

of CALLAHAN & KOSARAJU.

THM (WSPD):

Let $E \subset \mathbb{R}^n$, $\#(E) = N$.

Then we can partition $E \times E \setminus \text{Diagonal}$ into Cartesian products $E'_\nu \times E''_\nu$, $\nu = 1, \dots, \nu_{max}$

such that

(a) $\text{DISTANCE}(E'_\nu, E''_\nu) \geq 10^3 \cdot [\text{DIAM}(E'_\nu) + \text{DIAM}(E''_\nu)]$ for each ν

and

(b) $\nu_{max} \leq CN$,

where C depends only on the dimension n.

Moreover, an EFFICIENT ALGORITHM computes the $E_\nu' \times E_\nu''$.

In particular, we can produce "REPRESENTATIVES" $(x_\nu', x_\nu'') \in E_\nu' \times E_\nu''$ for all ν using at most $CN \log N$ computer ops.

DEPENDS ONLY ON THE DIMENSION n

WE WILL SEE IN A LATER LECTURE HOW THE WSPD LETS US COMPUTE $\|f\|_{Lip}$ TO WITHIN 1%.

THE WSPD WILL BE USED TO PROVE SEVERAL OF OUR RESULTS ON C^m AND $L^{m,p}$ INTERPOLATION.

WE WON'T EXPLAIN THE ALGORITHM OF CALLAHAN - KOSARAJU TO COMPUTE THE WSPD.

INSTEAD, WE SHOW HOW TO OBTAIN A WSPD USING WHITNEY CUBES.

HERE GOES :

CONSTRUCTING

A

WSPD

FOR

$E \times E \setminus$ DIAGONAL

LET $Q^0 \supset E$ BE A LARGE CUBE.

WE PERFORM A DECOMPOSITION

OF $Q^0 \times Q^0 \setminus$ DIAGONAL

INTO WHITNEY CUBES.

WE START WITH THE CUBE $Q^0 \times Q^0$,

AND WE STOP BISECTING $Q' \times Q''$

WHENEVER

$$\text{DISTANCE}(Q', Q'') \geq 10^3 \cdot [\text{DIAM}(Q') + \text{DIAM}(Q'')]$$

LET $Q'_\nu \times Q''_\nu$ $(\nu = 1, \ldots, \nu_{MAX})$

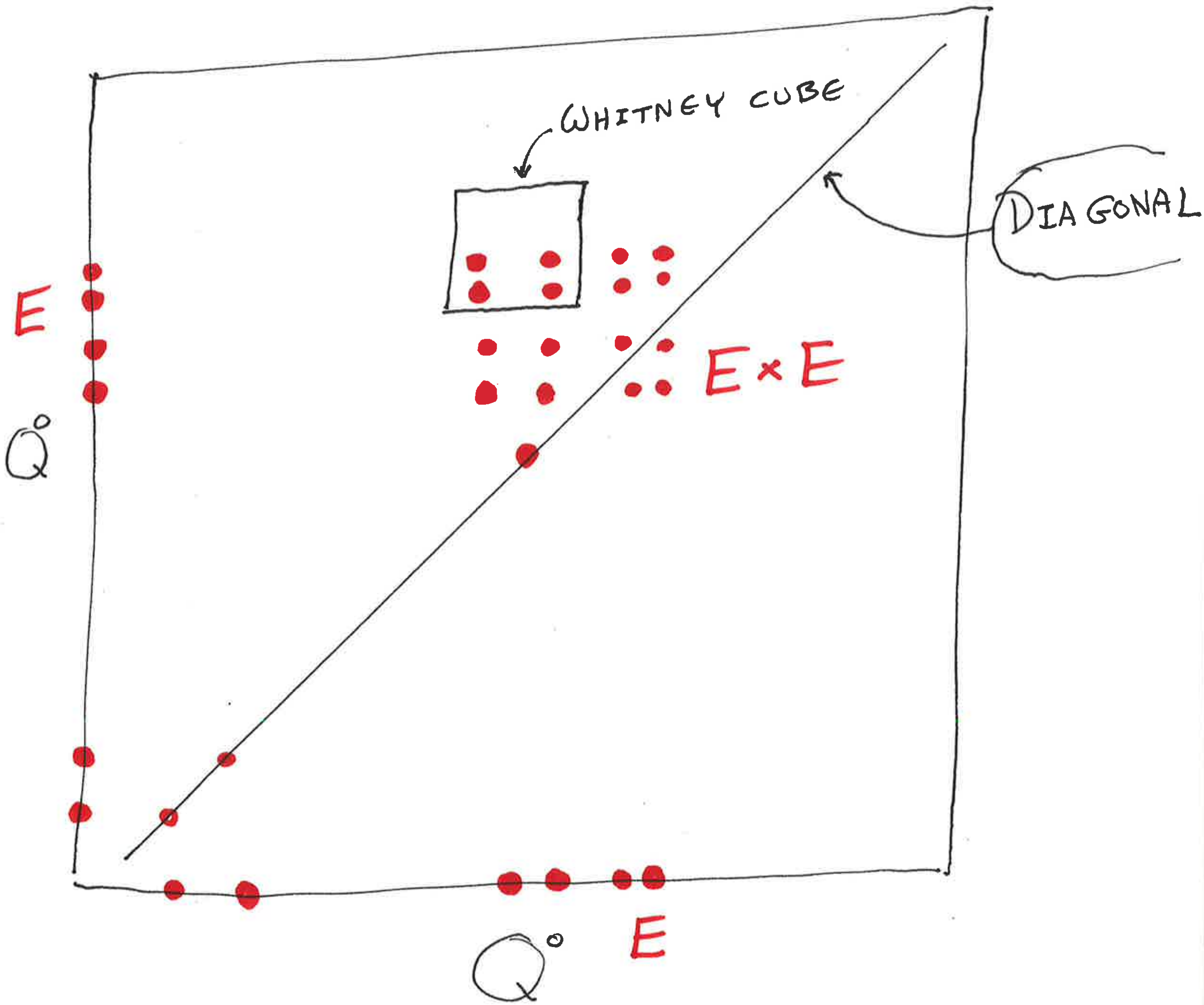
BE THE WHITNEY CUBES

THAT INTERSECT $E \times E$.

OUR CARTESIAN PRODUCTS $E'_\nu \times E''_\nu$

ARE SIMPLY THE SETS

$$(E \times E) \cap (Q'_\nu \times Q''_\nu)$$


WHITNEY CUBE
DIAGONAL
E
$\overset{\circ}{Q}$
$E \times E$
$\overset{\circ}{Q}$
E

CLEARLY,

$$\boxed{\begin{array}{l} E \times E \setminus \text{DIAGONAL} \\ \text{IS PARTITIONED INTO} \\ \text{THE } E_\nu' \times E_\nu'' \quad (\nu = 1, \dots, \nu_{MAX}) \end{array}}$$

AND

$$\boxed{\begin{array}{l} \text{DISTANCE}\,(E_\nu', E_\nu'') \geq 10^3 \left[\text{DIAM}\,(E_\nu') + \text{DIAM}\,(E_\nu'') \right] \\ \text{FOR EACH } \nu. \end{array}}$$

ONE CAN SHOW THAT

ν_{MAX} (THE NUMBER OF PRODUCTS $E'_\nu \times E''_\nu$)

IS AT MOST CN, where

$N = \#(E)$

and C depends only on the dimension n.

So we have found a

Well-Separated Pairs Decomposition.

I HOPE THIS CONVEYS

SOME HINT

OF THE POWER OF THE

WHITNEY / CALDERÓN - ZYGMUND

DECOMPOSITION.

WE'LL SEE MORE IN THE

LATER TALKS.

Thank you!